RESEARCH

# Algebra in the Revision

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**A major emphasis of the revision is to make the foundations of algebra currently in the curriculum more visible to teachers and students, to expand this work in the context of students' study of number and operations, and to deepen the focus on patterns, functions, and change by creating a consistent strand in Grades K–5.**

Algebra is a multifaceted area of mathematics content, and various schema have been proposed for classifying that content (e.g., Kaput, 1998, 1999; Usiskin, 1988). These references, as well as the NCTM's Principles and Standards, suggest several central aspects of algebra (these are related, overlapping categories): a) generalizing and formalizing patterns; b) representing and analyzing the structure of number and operations; c) using symbolic notation to express functions and relations; d) representing and analyzing change. In the Investigations revision, we have addressed these four aspects in two major ways.

## Integration of Early Algebra into the Units That Focus on Number and Operations

The materials highlight the generalizations about number and operations students frequently observe. Teachers learn to help students articulate these generalizations and challenge them to consider the questions: Does this generalization apply to all numbers (in the domain under consideration)? Why does it work? How do you know? Throughout Grades 1–5, students articulate, represent, investigate, and justify general claims. In each of the number and operations units, an essay, Algebra Connections in This Unit, highlights several generalizations and includes examples of how students think about and represent them. Investigation and discussion of some of these generalizations are built into unit sessions; at other times, "Algebra Notes" alert the teacher to sessions in which these ideas are likely to arise. For example, in Grade 2, students consider whether the order of terms affects the sum in addition problems or the diff erence in subtraction. In Grade 3, students discuss the generalization underlying the equivalence of subtraction expressions as in the equation, 104 – 78 = 106 – 80. In Grade 5, students justify why halving one factor and doubling the other in multiplication results in the same product (e.g., 65 x 24 = 130 x 12 = 1560).

## A Complete K–5 Strand on Patterns, Functions, and Change

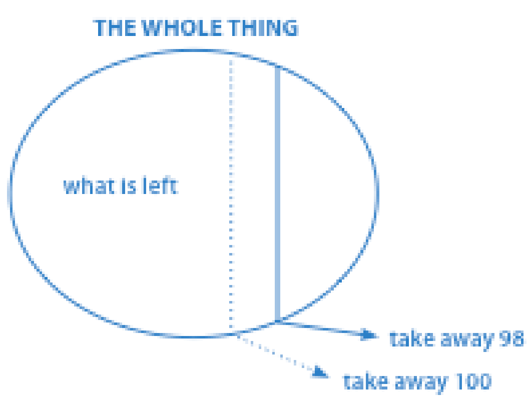
We have created a coherent K–5 strand that starts with repeating patterns and number sequences in Grades K and 1, connects to functional relationships beginning in Grade 2, and focuses on linear and nonlinear change in Grades 3–5. Students study relationships that follow rules (such as the relationship between the number of windows in a building and the number of fl oors if the building has a fi xed number of windows per floor) and relationships that do not follow rules (such as the relationship between temperature and time in Grade 3 and between plant growth and time in Grade 4). They work extensively with ways of representing these relationships: in words, with numbers, with tables and graphs, and (starting in Grade 4) with symbolic notation. These units reinforce and connect with work in other units on multiplication, ratio, area, volume, and graphing.

## EXAMPLES FROM THE CLASSROOM

To provide a flavor of the kind of work students do in the revised curriculum, here are two examples from field-test classrooms that fall under the first category: making and justifying general claims about number and operations.

### Grade 4: Is It 2 More or 2 Less?

In a Grade 4 class, students were trying to explain why the result of the subtraction, 145 – 98, is two more, not two less, than the result of the subtraction, 145 – 100. The context for the problem was related to a science project in which the class weighed apples in grams as they dried out. To address the subtraction problem, Brian drew a closed shape representing an "apple" divided into two parts, then used it to show what would happen if the part that is "taken away" is diminished while the whole stays the same. "See, this is the apple at first," he explained. "And you take some away [the part to the right of the dotted line] and have some left [to the left of the dotted line]. Then you take away 98 grams instead, so it's over here [the part to the right of the solid line is now the part that is subtracted; left of the solid line is what remains]."



With the presence of this picture to focus the discussion, more students joined in, using the representation not only to reason about the particular numbers, but to state and justify a more general claim. Rebecca said, "It's like you have this big hunk of bread and you can take a tiny bite or a bigger bite. If you take away smaller, you end up with bigger." Then Max stated: "The less you subtract, the more you end up with, AND in fact the thing you end up with is exactly as much larger as the amount less that you subtracted."

**Grade 2: Switching Around the Numbers**

But what about students in the primary grades? Aren't they "concrete" thinkers? In fact, young students, too, notice regularities about the work they do as they count, compare quantities, and learn about addition and subtraction. Here is an example from a second grade. The teacher asked the students to generate combinations of two addends with a sum of 25. As they listed these on chart paper, students soon noticed—as they had before in their computation work—that they could "switch around" the numbers and still get the same sum, for example, if 23 + 2 = 25, then 2 + 23 = 25. In fact, the teacher had in mind that this idea would come up during this activity and had planned follow-up questions. She asked, "Suppose I asked . . . if you could prove that or explain it better to me . . . if we take the 2 and put it fi rst, do we still get 25?" Nikki demonstrated with a stack of 23 cubes and a stack of 2 cubes. She moved the 2-cube stack rapidly and repeatedly from the right side to the left side of the 23-cube stack. "It doesn't matter," she said, "if you keep on just switching it around, it will still make 25 . . . you're not taking away or adding to it . . . it will still be the same number." Again, in this example, the use of a representation that embodies the operation enables the students to reason about the general claim. Although

Nikki is holding particular quantities—23 and 2—her reasoning applies to any pair of numbers. Once all the students seemed quite convinced that the order of any pair of numbers in an addition expression could be changed without changing the sum, the teacher asked the students if the same is true for subtraction. From her experience with these ideas, the teacher knew not to assume that students thought that the "switch around" rule applies only to addition. Students thought about her question for a few minutes, then several students offered their ideas, using 7 – 3 and 3 – 7 as an example.

Nikki: If you have 3 take away 7, but 3 doesn't have 7. . . . You can only take away 3 to make zero.

Alita: You can't use the 3 because after you use the 3—3, 2, 1, 0, 0, 0. . . the zero's going to keep on repeating itself. Edward: It wouldn't be zero. It would be negative 4 . . . That means you're going lower. If you're going lower than zero, that means negative 1, negative 2, negative 3 . . . .

Although these students did not yet have all the number experience necessary to understand this idea, the teacher noticed that they were making important observations about the diff erences between the properties of addition and the properties of subtraction. She planned to return to this discussion as other opportunities arose—for example, can the order of the numbers be changed in an addition expression with more than two addends without aff ecting the sum?

## ALGEBRA FOR ALL STUDENTS

The work of generalizing and justifying in the elementary classroom has the potential of enhancing the learning of all students. The teachers with whom we have collaborated for several years have realized this potential in their classrooms. Teacher collaborators report to us that students who tend to have diffi culty in mathematics become stronger mathematical thinkers through this work. As one teacher wrote, "When I began to work on generalizations with my students, I noticed a shift in my less capable learners. Things seemed more accessible to them." When the generalizations are made explicit—through language and through spatial representations used to justify them—they become accessible to more students and can become the foundation for greater computational fl uency. Furthermore, the disposition to create a representation when a mathematical question arises supports students in reasoning through their confusions. Brian (in the Grade 4 example above), a tentative learner in mathematics, created a representation that illuminated an important idea. In the second grade classroom, in an urban center with a historically large proportion of underachieving students, a range of students off ered important ideas about how addition is and subtraction is not commutative.

At the same time, students who generally outperform their peers in mathematics find this content challenging and stimulating. The study of number and operations extends beyond efficient computation to the excitement of making and proving conjectures about mathematical relationships that apply to an infinite class of numbers. A teacher explained, "Students develop a habit of mind of looking beyond the activity to search for something more, some broader mathematical context to fit the experience into." In the fourth grade example above, Max, one of the most mathematically successful students in the class, listened carefully to his classmates' explanations and then enjoyed the challenge of formulating a precise statement of the generalization. And Edward (in the Grade 2 example), who knew more about numbers than his peers, was able to seed the conversation with a new idea about numbers below zero.

## EARLY ALGEBRA IS FUNDAMENTAL

Underlying these kinds of discussions are what one of our mathematician advisors calls "foundational principles"—principles that connect elementary students' work in arithmetic to later work in algebra. For example, the idea explored by the fourth graders (the less you subtract, the more you have left) can be represented as, "If $a – b = c$, then $a – (b – x) = c + x$," or, more concisely, "$a – (b – x) = (a – b) + x$." A discussion among middle schoolers similar to that in the 4th grade example could provide an opportunity to consider why the associative property does not apply to subtraction, and to articulate a rule that does. The second graders do not yet have the experience with negative numbers to allow them to completely make sense of $3 – 7$, but they are nevertheless engaged in reasoning about foundational ideas, in this case, that addition is commutative, but subtraction is not: $a + b = b + a$, but $c – d \neq d – c$. In later years, they will come to see that there is a regularity here, that if $c – d = a$, then $d – c = -a$, or $c – d = - (d – c)$.

For most adults, notation such as the use of variables, operations, and equal signs is the chief identifying feature of algebra. Although we do introduce symbolic notation in Grade 4, the notation is not the focus of activity in Grades K–5. Underlying the notation are ways of reasoning about how the operations work. This reasoning—about how numbers can be put together and taken apart under diff erent operations or about relationships between two changing quantities—not the notation, is the central work of elementary students in algebra.

In the course of our work to integrate the foundations of algebra into the Investigations curriculum and through the insights of our fi eld-test teachers and the thinking of their students, we have concluded that work in early algebra is fundamental to the experience of young students. In summary:

1. Early algebra is not an add-on. The foundations of algebra arise naturally throughout students' work on number, operations, patterns, and through noticing how one thing changes in relation to another. This work anchors students' concepts of the operations and underlies greater computational flexibility.
2. Algebra as generalized arithmetic provides openings for working on reasoning and proving.
3. Algebra provides the opportunity to learn about the power of representation as a basis for mathematical reasoning.
4. Working on the algebraic underpinnings of arithmetic is one way of engaging the range of learners in mathematical thinking.

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