Developing Computational Fluency with Whole Numbers in the Elementary Grades

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When the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) was published, one of the parts of the document that received a great deal of attention was the chart of what should receive "increased emphasis" and "decreased emphasis." The "decreased emphasis" on paper-and-pencil computation resulted in a range of reactions: it was misinterpreted, oversimplified, embraced, and rejected. In developing *Principles and Standards for School Mathematics* (NCTM, 2000), the writers thought hard about the issues of computational fluency -- how to balance the need for both skills and understanding, how to make sure students develop both procedural competence and mathematical reasoning: "Developing fluency requires a balance and connection between conceptual understanding and computational proficiency. On the one hand, computational methods that are over-practiced without understanding are often forgotten or remembered incorrectly... On the other hand, understanding without fluency can inhibit the problem-solving process (p. 35)." As we look at what this new document -- based on the 1989 *Standard*s but benefiting from a decade of research and practice -- has to say about computational fluency, it is important to revisit the problem we are trying to solve -- the lack of computational fluency that is well documented among U.S. students.

What is the Problem?

Recently, during a visit to a third grade classroom, I watched Eleana adding 112 and 40. She had written the numbers like this, misaligning the columns:

$$\begin{array}{r} 112 \\ +\ 40 \\ \hline 512 \end{array}$$

She called me over because the answer didn't seem right to her, but she *knew* she had done the steps correctly. So I said something like, "What if the problem was 110 plus 40? Try to figure that out in your head." She figured it out by starting at 110 and counting on by 10's (120, 130, 140, 150). Then she knew that she needed to add on the 2 and solved the problem easily.

In another third grade class, David solved a multiplication problem like this:

$$\begin{array}{r} {\scriptstyle 2} \\ 57 \\ \times\ 4 \\ \hline 288 \end{array}$$

He had multiplied the 7 and 4, getting 28, put down the 8 and "carried" the 2, then added the 2 to the 5 *before* multiplying by 4, giving him an answer of 288. I said to him, "So, is there a problem you know that's close to 57 x 4 that could help you with this problem?" He shook his head. I said, "Do you know what 50 x 4 would be?" "Um, sure -- two 50's would be 100, so it's 200." Then I asked him what part of the problem he had left to solve. He said he needed to solve 7 x 4. He finished solving the problem, adding the two subproducts in his head. For this student, 57 x 4 should have been an easy mental problem.

Eleana and David are students who have some good ideas about number relationships and operations, but they have not learned to apply these ideas sensibly in solving problems. They draw on partially remembered techniques they have been taught, but they do not relate these procedures to the meaning of the numbers or the properties of the operations they are carrying out. David was very proud of the multiplication procedure he had learned and wanted me to give him "hard problems" so he could demonstrate his competence in mathematics. It is critical all students do develop competence in mathematics. The route of learning traditional procedures by rote, without understanding, has severely undeserved our students. There are decades of data showing that US students have learned to compute simple problems successfully through traditional instruction but are not able to use this learning as a basis for solving more complex problems.[2]

Connecting Understanding and Procedures

Eleana's and David's work are typical examples of using procedures without understanding. Some might say that they simply need to learn the rules better in order to succeed. However, the students' behavior reveals that simply correcting the procedure, as if they were programmable calculators, is not the issue. They are not looking at the whole problem and using what they know about numbers and operations to reason about the answer. They do have knowledge they could draw on, but they do not think to use it. The orientation of these students in mathematics class is to try to remember steps rather than to build on what they know in order to make sense of the problem. Think for a minute about the following problem:

$$\begin{array}{r} 1002 \\ -\ 998 \\ \hline \end{array}$$

If a fourth grade student solves this problem competently by using a multi-step procedure, it is time to worry about what the student knows about whole number operations. A student who has achieved computational fluency should be able to solve this problem mentally without hesitation by simply noticing the relationship of the two numbers (try 102 - 98 for a younger student). The student should clearly see that 998 is 2 less than 1000 and that 1002 is 2 more than 1000, so that the difference is 4.

Students do need procedures to solve problems, but procedures alone are not enough. Both for their current work in the elementary grades and for future work in mathematics, students must learn about the structure of the base ten system and about the properties of operations as the foundation of computational fluency. Learning whole number computation is one of the key contexts in which students engage deeply with these two important bases of mathematics. David, when probed, does know something about the properties of multiplication. With help, he recognizes that 57 x 4 can be broken into two subproblems, 50 x 4 and 7 x 4, and that adding the two products together will give him the answer to the problem -- an application of the distributive property. However, he currently has not integrated these two pieces of knowledge -- the computation procedure and understanding of properties of multiplication. He does not recognize that 288 (which would be closer to 4 x 70 than to 4 x 50) is not a sensible answer. Dowker (1992) quotes Hadamard (1945): "Good mathematicians, when they make errors, which is not infrequent, soon perceive and correct them." Dowker goes on, "To the person without number sense, arithmetic is a bewildering territory in which any deviation from the known path may rapidly lead to being totally lost (p. 52)." Our job as teachers is to help students connect procedures, properties of operations, and understanding of place value, rather than having them learn these as separate, compartmentalized pieces of knowledge.

Some of the traditional algorithms historically taught in US schools make it difficult for students to make these connections. Let us look at 57 x 4 again. The traditional algorithm splits up the first subproduct, 28, into 20 and 8, leaving the 8 under the line and putting the 20 (which is written as a 2, so it no longer looks like 20) above the tens column. The algorithm suggested by David's approach keeps the result of each subproblem intact. The distributive property is much more visible as is the place value of all the numbers involved in the problem:

$$\begin{array}{r} 57 \\ \times\ 4 \\ \hline 200 \\ 28 \\ \hline 228 \end{array}$$

David's approach is sound mathematically, connects clearly to his understanding of the mathematical relationships in the problem, and provides a good foundation for solving more complex problems. Teaching that enables students to connect procedures, understanding of place value, and properties of operations is not simply a matter of using more transparent procedures and notation. Students must develop algorithms *while* they are developing their understanding of place value and of the operations, not separately. There is extensive evidence that, for many students, this kind of integrated learning of procedures and understanding leads to the development of algorithms that are different from those traditionally taught in the United States (see, for example, Hiebert et al., 1997, for a look at three major relevant research programs). Further, there is growing evidence that once students have memorized and practiced procedures without understanding, they have difficulty learning to bring meaning to their work (Hiebert, 1999).

On the other hand, efforts to do a better job of teaching for computational fluency have also had their pitfalls:

1. <u>Depending on mathematics manipulatives to do the teaching.</u> At times, manipulatives such as base ten blocks or Cuisenaire rods have been seen as a magic cure (Ball, 1992). When I taught mathematics methods courses to graduate students, adults in the class would commonly come to new insights about the structure of the number system as they used base ten blocks and mentally abstracted the structure of the blocks to higher numbers. It is easy to conclude from these kinds of experiences that just by using the blocks, students will automatically "see" the model of the base ten system. But the mathematical relationships are not "in" the blocks; rather, students can, with appropriate experience, build a model of the relationships through a variety of experiences including using such materials. As Hiebert and his colleagues (1997) point out, "Mathematical tools should be seen as supports for learning. But using tools as supports does not happen automatically. Students must construct meaning for them. This requires more than watching demonstrations; it requires working with tools over extended periods of time, trying them out, and watching what happens. Meaning does not reside in tools; it is constructed by students as they use tools (p. 10)."

2. <u>Sharing strategies for solving problems without examining them.</u> As classrooms focus more on student thinking, classroom teachers often experience great excitement as they see students developing and discussing mathematical ideas of their own -- an experience they may not have had in their own schooling. Sharing different strategies for solving a problem is one way to acknowledge and encourage multiple approaches to problems, to expose students to problem solutions different from their own, and to look for mathematical relationships. However, it is critical that students also learn to examine strategies -- to analyze them, critique them, and find

relationships among them. They need to develop the habit of explaining and justifying their solution strategies and the facility for analyzing incorrect procedures (Russell, 1999). The goal is not simply to come up with lots of strategies and compare them, but -- over time -- to identify and use general mathematical principles that lead to consistent, reliable solutions for a class of problems.

3. Stopping short of consolidation, generalization, and scaling to larger problems. At first, when students are developing their understanding of an operation, every problem is a new problem. They may experiment with several algorithms and procedures and gradually begin to match algorithms to the problem. For example, Jemelle, a second grader, might solve most addition problems using a left-to-right algorithm, adding the largest place first (Problem A in Figure 1), but might also notice that some problems lend themselves to creating an equivalent problem that is easier to solve (for example, Problem B in Figure 1).

Figure 1. Alternative solution paths a child might provide for addition problems.

43 + 28	**52+ 38**
40 + 20 = 60	**52 + 38 = 50 + 40 = 90**
3 + 8 = 11	
60 + 11 = 71	
Problem A	*Problem B*

However, many students need help consolidating their good ideas about solving problems so that they have consistent, reliable, well-rehearsed methods for solving a class of problems. This means that once students have developed algorithms for problems that they understand and can perform easily, they also need support in: 1) practicing them so that they are fluent; 2) developing notation that is clear and easy to follow; and 3) generalizing their algorithms to larger problems. We have often moved to the practice and consolidation step too soon. For example, Liping Ma (1999) points out that the Chinese teachers she interviewed placed much more importance on multiplication of 2-digit numbers than of larger numbers: "According to them, the more solid the first and primary learning is, the more support it will be able to contribute to later learning of the concept in its more complex form (p. 53)." Nevertheless, once this solid base is established, teachers need to help students generalize and consolidate their methods.

Jemelle needs to practice each of her methods so that she can solve such problems easily (this includes knowing her addition combinations well). She needs to discuss with the teacher and with fellow students which method she chooses for different problems and why. As she moves into third and fourth grade, she needs to develop clear, simple ways of notating her work and to gain experience in applying these methods to larger numbers. So, for example, to add 1248 and 3992, she needs to have sound ideas about the relationship of 3992 to 4000, so that she can easily recognize that an equivalent problem is 1240 + 4000. At this point, too, students can be introduced to some of the algorithms historically taught in the U. S. so that they become familiar with these commonly used algorithms, understand them, and can choose to use them. However, shortcut notations that obscure place value, such as those used in the traditional "carrying" algorithm, must be introduced with great care, so that students do not lose the meaning of the problem as a whole through rote manipulation of individual digits.

What is Computational Fluency?

Fluency, as we use it here, includes three ideas: efficiency, accuracy, and flexibility:

- *Efficiency* implies that the student does not get bogged down in too many steps or lose track of the logic of the strategy. An efficient strategy is one that the student can carry out easily, keeping track of subproblems and making use of intermediate results to solve the problem.
- *Accuracy* depends on several aspects of the problem-solving process, among them careful recording, knowledge of number facts and other important number relationships, and double-checking results.
- *Flexibility* requires the knowledge of more than one approach to solving a particular kind of problem, such as two-digit multiplication. Students need to be flexible in order to choose an appropriate strategy for the problem at hand, and also to use one method to solve a problem and another method to double-check the results.

Fluency demands more of students than does memorization of a single procedure. Fluency rests on a well-built mathematical foundation with three parts: (1) an understanding of the meaning of the operations and their relationships to each other -- for example, the inverse relationship between multiplication and division; (2) the knowledge of a large repertoire of number relationships, including the addition and multiplication "facts" as well as other relationships, such as how 4 X 5 is related to 4 X 50; and (3) a thorough understanding of the base ten number system, how numbers are structured in this system, and how the place value system of numbers behaves in different operations -- for example, that 24 + 10 = 34 or 24 X 10 = 240.

Assessing Fluency

Let's look at some examples of students in a grade 5 classroom working on the division problem:

159/13

Cara thought about the problem as, "how many 13's are in 159?" She knew that ten 13's is 130, that she then had 29 remaining in the dividend, and that she could take two 13's out of 29, giving her an answer of 12 with a remainder of 3.
Armand counted by 13's until he reached 52. He then added 52's until he got as close to 159 as possible (52 + 52 + 52 = 156). He knew there were four 13's in each of the 52's, so he had twelve 13's , with a remainder of 3.

Malaika subtracted 13's from 159 until she couldn't subtract any more. She explained, "I kept taking away. I made a mistake, but I checked and fixed it, and I kept taking it away. I counted how many I took away. It was 12, and I had a remainder 3 because I couldn't take away any more 13's."

What do these three methods reveal about the computational fluency of these three students? Each of the students knows something about what division is and its relationship to other operations: Cara uses multiplication, Armand uses addition, and Malaika uses subtraction correctly to solve the division problem. Cara and Armand display more knowledge of the properties of division: each of them is able to break apart the original problem into parts and then to put the parts in correct relationship to each other. Distributivity seems to underlie both of their solutions. Cara's solution suggests thinking of the problem as (130 ⁄ 13) + (29 ⁄ 13), while Armand's solution suggests splitting up the problem differently as (52 ⁄ 13) + (52 ⁄ 13) + (52 ⁄ 13) + (3 ⁄ 13).

However, these students have very different needs. Cara is identifying a large, easily solvable subproblem (130 ⁄ 10), using her knowledge of the structure of the number system to choose it. She needs to pursue and, eventually, consolidate this approach so that she can use it efficiently and consistently. She may also need help generalizing this approach to larger numbers. Does she know the effect of multiplying by 20? For example, if she were dividing 400 by 18, would she know that 20 x 18 = 360?

Armand is also beginning to take apart the problem into manageable subproblems, but he seems to rely on counting by the divisor to find these subproblems. In this problem, it happened that he encountered a convenient number, 52, that was easy to combine. What if this didn't occur? Armand needs work on using the structure of the base ten system to help him. Does he know the effect of multiplying by 10?

Malaika, as a fifth grader, should cause a teacher great concern. While she does have some understanding about the operation of division, subtracting 13's is a tedious, error-prone, and inefficient way to solve this problem. What does she know about the relationship between multiplication and division? What does she know about multiplying by 10? These are areas in which she needs considerable work.

Here are some questions to consider as you assess computational fluency among your students:

- Does the student know and draw on basic facts and other number relationships?
- Does the student use and understand the structure of the base ten number system -- for example, does the student know the result of adding 100 to 2340 or multiplying 40 x 500?
- Does the student recognize related problems that can help with the problem?
- Does the student use relationships among operations?
- Does the student know what each number and numeral in the problem means (including subproblems)?
- Can the student explain why the steps he or she uses works?
- Does the student have a clear way to record and keep track of his or her procedure?
- Does the student have a few approaches for each operation so that a procedure can be selected for the problem (as Jemelle was learning to do)?

As the Standards for Grades 3-5 in *Principles and Standards* (NCTM, 2000) state: "Students exhibit computational fluency when they demonstrate flexibility in the computational methods they choose, understand and can explain these methods, and produce accurate answers efficiently. The computational methods that a student uses should be based on mathematical ideas that the student understands well, including the structure of the base-ten number system, properties of multiplication and division [and, of course, of the other operations, as well], and number relationships (p. 152)."

Teaching for Computational Fluency

Teaching for both skill and understanding is crucial -- these are learned together, not separately. But teaching in a way that helps students develop both mathematical understanding and efficient procedures is complex. It requires that teachers understand the basic mathematical ideas that underlie computational fluency, use tasks in which students develop these ideas, and recognize opportunities in students' work to focus on these ideas.

Many of us learned mathematics as a set of disconnected rules, facts, and procedures. As mathematics teachers, we then find it difficult to recognize the important mathematical principles and relationships underlying the mathematical work of our students. Those responsible for the professional development of teachers are increasingly coming to understand the need for long-term opportunities for teachers to deepen their understanding of mathematical content. There are now a variety of professional development structures and sets of materials that can provide the basis for such long-term work in which, through intensive institutes or regular, ongoing school-year seminars, teachers immerse themselves in both mathematical content and in learning about children's mathematical thinking. We have examples of how teachers who have participated in such experiences engage their students with fundamental mathematical ideas. For example, Virginia Brown, a third grade teacher, recognizes the opportunity to develop her students' understanding of commutativity (Brown, 1996), while Joanne Moynahan (Moynahan, 1996) helps her sixth graders use what they know about the inverse relationship of multiplication and division to understand how to use these operations with fractions.

This kind of teaching leads students not to *memorizing,* but to the development of *mathematical memory* (Russell, 1999). Important mathematical procedures cannot be "forgotten over the summer," because they are based in a web of connected ideas about fundamental mathematical relationships. As Ma (1999) says in her study of Chinese and American teaching of elementary mathematics: "Being able to calculate in multiple ways means that one has transcended the formality of the algorithm and reached the essence of the numerical operations -- the underlying mathematical ideas and principles. The reason that one problem can be solved in multiple ways is that mathematics does not consist of isolated rules, but connected ideas. Being able to and tending to solve a problem in more than one way, therefore, reveals the ability and the predilection to make connections between and among mathematical areas and topics (p. 112.)" Attaining computational fluency is an essential part of students' elementary mathematics education. It is much more than the learning of a skill; it is an integral part of learning with depth and rigor about numbers and operations.

Footnotes

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[2] See Hiebert (1999) for a detailed history of the inadequacy of traditional mathematics instruction.

[3] Adapted from videotaped examples in Russell et al. (1999).

References

Ball, D. (1992). Magical hopes: Manipulatives and the reform of math education. *American Educator,* 16 (1), 14-18, 46-47.

Brown, V. M. (1996). Third graders explore multiplication. In *What's happening in math class? Vol. 1: Envisioning new practices through teacher narratives,* edited by D. Schifter, pp. 18-24. New York: Teachers College Press.

Dowker, A. (1992). Computational strategies of professional mathematicians. *Journal for Research in Mathematics Education,* 23 (1), 45-55.

Hadamard, J. (1945). *The psychology of invention in the mathematical field.* Princeton, NJ: Princeton University Press.

Hiebert, J. (1999). Relationships between research and the NCTM Standards. *Journal for Research in Mathematics Education,* 30 (1), 3-19.

Hiebert, J., Carpenter, T.P., Fennema, E., Fuson, K., Wearne, D., Murray, H., Olivier, A., & Human, P. (1997). *Making sense: Teaching and learning mathematics with understanding.* Portsmouth, NH: Heinemann.

Ma, L. (1999). *Knowing and teaching elementary mathematics.* Mahwah, NJ: Erlbaum.

Moynahan, J. (1996). Of-ing fractions. In *What's happening in math class? Vol. 1: Envisioning new practices through teacher narratives,* edited by D. Schifter, pp. 24-36. New York: Teachers College Press.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: NCTM.

National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics.* Reston, VA: NCTM.

Russell, S. J. (1999). "Mathematical reasoning in the elementary grades." In *Developing mathematical reasoning in grades K-12, 1999 Yearbook of the National Council of Teachers of Mathematics*, edited by L. V. Stiff and F. R. Curcio, pp. 1-12. Reston, VA: NCTM, 1999.

Russell, S. J., Smith, D., Storeygard, J., & Murray, M. (1999). *Relearning to teach arithmetic: Multiplication and division.* Parsippany, NJ: Dale Seymour Publications.

Reference
Russell, Susan Jo. (May, 2000). Developing Computational Fluency with Whole Numbers in the Elementary Grades. In Ferrucci, Beverly J. and Heid, M. Kathleen (eds). Millenium Focus Issue: Perspectives on Principles and Standards. *The New England Math Journal*. Volume XXXII, Number 2. Keene, NH: Association of Teachers of Mathematics in New England. Pages 40-54.