Investigations 3: Making Sense of and Persevering with the Mathematical Practices

BY DEBORAH SCHIFTER AND SUSAN JO RUSSELL

Students' experiences in the elementary grades are critical to how they come to view mathematics. In these grades, does mathematics invite them in or shut them out? Do students come to think of mathematics as intriguing and engaging or as boring and unapproachable? Do they learn that they can have mathematical ideas? Do they learn to willingly tackle unfamiliar problems?

Developing a productive orientation to mathematics is fundamentally about the practices of the discipline. Over the years, several policy documents have identified essential aspects of what it means to do mathematics productively. The National Council of Teachers of Mathematics' *Principles and Standards for School Mathematics* (2000) defined five "process standards": problem solving, reasoning and proof, communication, connections, and representation. The National Research Council's report, *Adding It Up* (2001), defined five strands of mathematical proficiency: adaptive reasoning, strategic competence, conceptual understanding, procedural fluency, and productive disposition. Each of these documents has a particular take on how to define what is important to pay attention to in the doing of mathematics.

More recently, the eight Mathematical Practices spelled out in the *Common Core State Standards for Mathematics* (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010; Illustrative Mathematics, 2014) provide a useful and comprehensive framework for thinking about what students need to learn about engaging in this discipline. The third edition of the *Investigations* curriculum uses this framework to make more explicit the practices that have always been embedded in the materials and includes essays, sidebars, and assessments to help teachers understand more deeply the potential for student engagement.

STANDARDS FOR MATHEMATICAL PRACTICE

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

TAKING THE MATHEMATICAL PRACTICES SERIOUSLY AS PART OF INSTRUCTION

In planning for a significant focus on the CCSSM Mathematical Practices in the third edition of *Investigations*, the authors were committed to making these practices explicit and central in the work students do. Each practice is many-faceted and requires deliberate attention in order to incorporate it as an ongoing, significant part of students' mathematics experience. Four principles underlie how the Mathematical Practices are incorporated into the *Investigations 3* curriculum:

1. **Mathematical Practices comprise the ways in which one approaches mathematics.** Mathematical Practices have to do with one's orientation to engaging in mathematics—what you expect, what you look for, what you pay attention to. These practices require action and reflection. The Mathematical Practices all begin with verbs: *make sense, persevere, reason, construct, critique, model, use, attend, look for,* and *express*.

2. **Mathematical Practices are always employed in the context of mathematics content.** Students can't engage in Mathematical Practices apart from being engaged in mathematics content. You can't learn how to make sense of problems, for example, if you're not working on a problem that you need to make sense of. You can't reason abstractly and quantitatively without reasoning *about* mathematics content, or construct and critique mathematical arguments unless the arguments are about significant mathematical ideas. There can not be separate lessons, without content, for the Practices. Students can only learn to engage in these Practices while they are *doing mathematics*.

3. **Mathematical Practices don't just happen—they require focused instruction, just as mathematics content does.** Students don't automatically learn to make sense of problems any more than they automatically understand the base ten-number system. Just as students develop understanding of mathematics content through multiple, focused mathematics experiences, they develop an understanding of how to engage in mathematical practices through targeted, intentional, planned instruction. It is not sufficient to post a list of practices on the wall or have students make a check on a list when they think they are engaging in a particular practice. Students must have many opportunities to learn how to engage in these practices.

4. **While Mathematical Practices are never entirely mastered, students' growth in these Practices should be assessed at each grade level as it develops within specific content areas.** Although Math Practices require the same kind of focused learning time that math content does, students are never finished with, for example, making sense of problems or using appropriate tools strategically. As they encounter new domains of mathematics, new problem types, and new tools, they will build on and expand their proficiency with these Practices. Nevertheless, students' growth can be documented and assessed. In fact, assessing how students engage in each is key to taking seriously the teaching and learning of the Practices.

WHAT DOES IT LOOK LIKE WHEN ELEMENTARY STUDENTS ENGAGE IN THE MATH PRACTICES?

The following three vignettes from *Investigations 3* classrooms illustrate what the Mathematical Practices, guided by the principles presented above, look like in action.

Example 1: Grade 1
In this classroom episode, students are working on representing addition of 2-digit numbers in the unit, *How Many Tens? How Many Ones?* The class was given the problem 41 + 23, and as a group, students describe to the teacher how to represent the numbers with cubes.
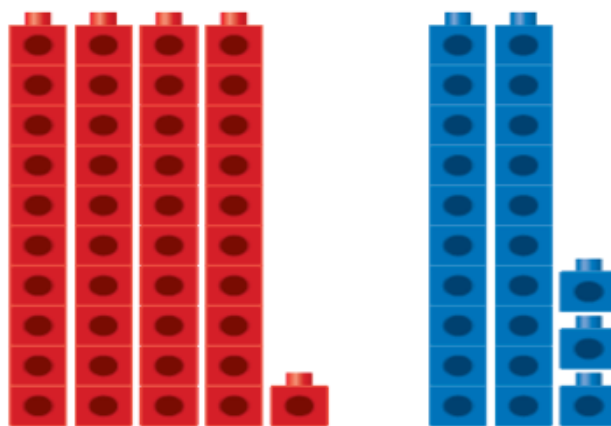

Students recount each set of cubes—10, 20, 30, 40, 41; 10, 20, 21, 22, 23—to double check.

The teacher then solicits students' ways of finding the total number of cubes. Maddie suggests joining all the cubes and counting them by 1s. Emilia starts with 41,

and then counts on by 1s, touching each blue cube as she counts. Jacob says his way is like Emilia's, but he counts by 10s—starting at 41, he counts 51, 61 (touching the blue towers)—and then counts the 1s (touching the single blue cubes)—62, 63, 64. The teacher points out the difference between Emilia's and Jacob's approaches: "So we can start at 41 and count on 23 by ones. We can also start at 41 and count on the tens, and then the ones."

When the class resumes sharing their strategies, Bruce says, "If you put all the tens together… can I do it?" Bruce uses the cubes to show the class how to group the tens from the two addends together. He counts the tens, and then the ones—10, 20, 30, 40, 50, 60, 61, 62, 63, 64.
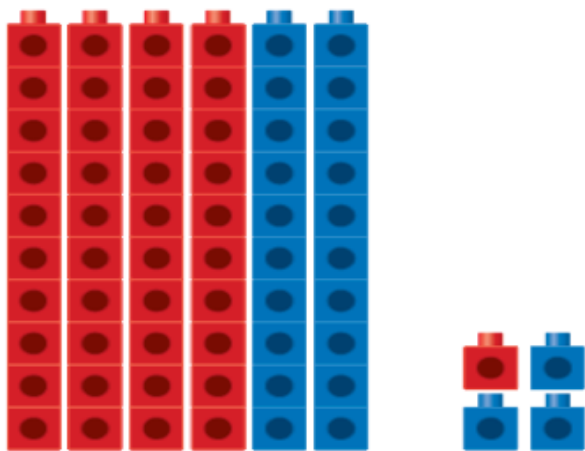
Allie: I did it the same way I solved the other ones. It's kind of like what Bruce did. You put the tens together and put the ones together, and that makes the new number. 6 tens and 4 ones is 64.

The teacher moves the cubes back to show 41 and 23 separately. Then she asks Allie to repeat her idea and show the class how it works with the cubes.

Although it rarely happens that students implement all eight Mathematical Practices in a single activity, it frequently happens that students are engaged in several Practices at the same time. In this vignette, students are engaged in at least four: MP1, MP2, MP7, and MP8.

MP1: Make sense of problems and persevere in solving them. MP1 is at play whenever students are challenged to reason through a problem. In fact, students are expected to be engaged in MP1 in all *Investigations 3* activities. In this Grade 1 classroom, solving the problem correctly and getting the answer of 64 is important, but another significant focus of the discussion is on

the variety of strategies students employed, that is, how they made sense of the calculation. Students are responsible not only for explaining their own approach, but also for following the reasoning of their classmates.

MP2: Reason abstractly and quantitatively.
At the beginning of the discussion, given the abstract expression, 41 + 23, the class works together to represent the problem with cubes. Maintaining connections between contexts and symbols is the intention of MP2. It is by holding in mind the abstraction (the numerical expression) along with a corresponding context (joining quantities of cubes) that students make meaning for symbols. In this lesson, Maddie states explicitly that solving 41 + 23 involves joining 41 cubes with 23 cubes and counting the total. With the representation of cubes as a context, students can reason quantitatively with the cubes and also think abstractly about breaking up the numbers in different ways—either counting on from 41 or first grouping the tens together and then counting. Furthermore, students use the cubes to explain their strategies to classmates.

MP7: Look for and make use of structure. Most of the students' strategies presented in this episode make use of the tens-and-ones structure of two-digit numbers. When counting on from 41, Jacob first counts on by tens, then by ones. Before they begin counting, Bruce and Allie group the tens together and group the ones together. Strategies that make use of the structure of tens and ones lead to greater efficiency.

MP8: Look for and express regularity in repeated reasoning. After Emilia and Jacob present their methods, the teacher says, "So we can start at 41 and count on 23 by ones. We can also start at 41 and count on the tens, and then the ones." She describes the two strategies in order to show how they are different: "counting on by ones" versus "counting on the tens and then the ones." In this way, students have a way to describe and categorize different strategies, recognizing what is consistently repeated in each as they apply them to different numbers and developing language for expressing each approach. Later, Allie, in particular, is engaged in MP8 when she notices she applied the same strategy in several problems: "You put the tens together and put the ones together, and that makes the new number." In her explanation, Allie does not refer to specific numbers, but describes a generalizable strategy for adding any pair of two-digit numbers.

Example 2: Grade 3

In the unit, Perimeter, Area, and Polygons, students measure the area of their footprints. Kathryn has traced her footprint on grid paper and has outlined some rectangles of square units inside the outline of her foot.
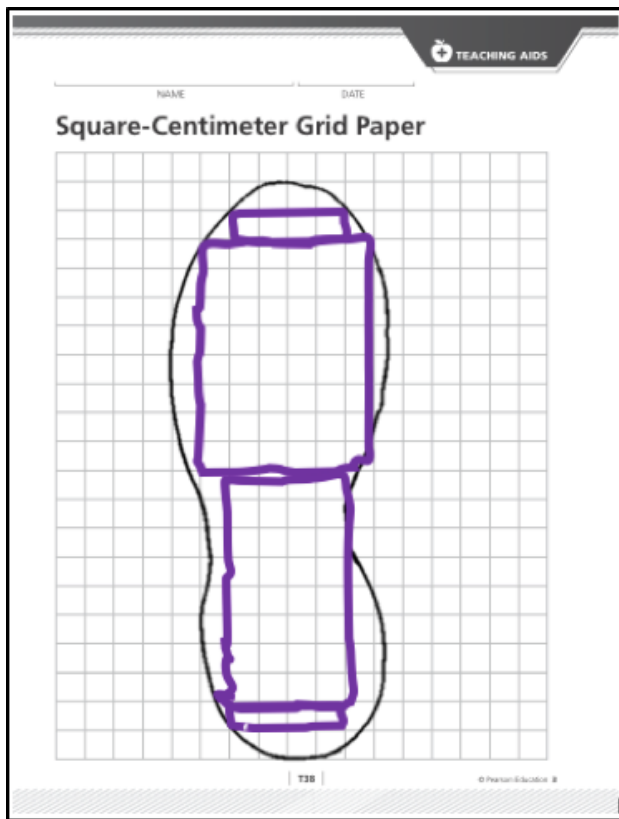
Kathryn: You're supposed to use multiplication, so I made some rectangles, but I can't get the edges. So I added up these rectangles, and that's the closest I can get.

Teacher: What do the rest of you think about Kathryn's method?

Deondra: I don't think you have to use multiplication. It works for rectangles, but this isn't a rectangle.

Cameron: If we leave out the edges, it's not the area; it's just part of the area.

Gina: You can count the extra squares around the edges of the rectangles.

Nancy: But it's still not perfect if you just count whole squares. There are all those little pieces. I don't get what you do with those. They're not squares, so they're not part of the area.

Keisha: Yes, they are. It's like with centimeters. You can have a half centimeter or a quarter centimeter. So you can have part square centimeters, too. If you don't count them, it's not the whole area.

Dwayne: You can pair up two pieces that are about a whole square.

Pilar: Even little bitty pieces, you can kind of say, well these 3 or 4 go together to make one square centimeter. I know it's not perfect, but it's closer than if you just ignore them.

As with any lesson in which students are challenged to apply their own reasoning, the students in this class are engaged in **MP1: Make sense of problems and persevere in solving them.** They make sense of the task, to measure their footprint, which provides them with the opportunity to understand more deeply the meaning of area. The vignette also illustrates two other Mathematical Practices, MP6 and MP3.

MP6: Attend to precision. After Kathryn presents her way of thinking about the problem—finding the number of squares in a set of rectangles contained within the boundary of her footprint—students describe ways to treat parts of the footprint that are not contained within Kathryn's rectangles. Once they agree that they need to include partial squares, Dwayne and Pilar offer strategies for determining the most precise measurement possible. This is one aspect of MP6.

MP3: Construct viable arguments and critique the reasoning of others. As students present their methods for finding the area of the footprint and attend to the ideas offered by their classmates, they engage with both parts of MP3. Note that critique does not mean to criticize. Rather, critiquing involves listening attentively to others' reasoning, following their ideas, and questioning or challenging when you don't understand or agree. The students in the first part of the discussion listen to Kathryn's method and explain why her strategy does not provide a precise measurement. As the discussion continues, students connect their ideas to what others have said and present their arguments for methods that better approximate the area of the footprint.

Example 3: Grade 5

In the unit, *Temperature, Height, and Growth,* Grade 5 students analyze patterns and rules, and use tables, graphs, and equations to model contexts. One of the contexts involves fictional animals from the imaginary planet Rhomaar where each animal is a certain height

at birth and then grows the same amount each year. In one class session, students complete a table showing the heights of the Krink and the Water Weasel. They are given the first four rows of the table and fill in the rest. In the next session, the class discusses what they notice about each animal and how the growth of the two animals compares, using the table to support their ideas.

Age (years)	Krink Height (cm)	Water Weasel Height (cm)
0 (birth)	1	15
1	6	20
2	11	25
3	16	30
4	21	35
5	26	40
6	31	45
7	36	50
8	41	55
9	46	60
10	51	65
15	76	90
100	501	515

Cecelia: Me and Lourdes noticed that the Water Weasel is taller than the Krink when they're born. The Krink is tiny, only 1 centimeter, but the Water Weasel is 15 centimeters.

Felix: But then they go up by the same amount each year.

Teacher: Which animal is growing faster? How can you see this in the table?

Charles: I think the Water Weasel is growing faster, because look at the numbers in the table. It's getting into the 40s and 50s.

Margaret: I think the Water Weasel is taller, but I don't think it's growing faster, because, look, it's just going by 5s, but the Krink is going by 5s, too.

Teacher: How do you know that? Who can elaborate on what Margaret and Charles are disagreeing about?

Yumiko: I can tell by looking down the two columns of the table. They're both going by 5s, so the Water Weasel doesn't get farther ahead.

Charles: I get what Margaret is saying. The Water Weasel is taller, but they just stay the same amount apart.

Avery: Look here at the table. They're always 14 centimeters apart. Look at any year and it's 14. It never gets more.

In addition to MP1, this vignette illustrates another two Mathematical Practices, MP4 and MP5.

MP4: Model with mathematics. The term modeling is used here to mean that students are identifying the mathematical elements of a situation (in this example, a fantasy situation) and representing them with mathematics, in this case, a table. They are abstracting the *mathematical* parts of the context and using that model, i.e., the table, to examine and interpret the relationships among the quantities.

MP5: Use appropriate tools strategically. Throughtout the elementary grades, students use a wide variety of mathematical tools. In the Grade 1 example above, cubes are tools that help students reason about addition and place value. In the Grade 3 example, students use a grid to measure area. In this Grade 5 vignette, tables are tools through which students examine the relationship between two variables. Later in this same discussion, students use expressions as a tool to model the situation.

> **Talisha:** For the Water Weasel, it's age times 5, but you have to add 15. [Talisha writes, "Water Weasel: Age × 5 = 15."]
>
> **Zachary:** When it's two years old, it's not just 2 times 5 centimeters, it's 2 times 5 centimeters plus the 15 it was when it was born.

The students are taking steps towards using an equation—e.g., Height = Age × 5 + 15—as a mathematical model of the relationship between the height of the Water Weasel and its age.

WHAT IS THE TEACHER'S ROLE IN SUPPORTING THE MATH PRACTICES?

In each of the examples above, the teacher plays a critical role in helping students to engage in the Mathematical Practices. In the first example, the teacher asks students to show their strategies with the cubes. Moving back and forth between the representation and the numbers supports students in reasoning abstractly and quantitatively (MP2) and formulating generalizable strategies (MP8). In the second example, the teacher invites students both to articulate their own arguments and to critique others' ideas (MP3). She focuses them on constructing ways to make their measurements more precise (MP6). In the final example, the teacher asks students to use a tool (MP5) to model a context (MP4). Her question, "Which animal is growing faster?", focuses them on the mathematical elements in the situation. See *From Principles to Actions* (NCTM 2014) for further discussion of teacher actions that support students' engagement with the Practices.

It is evident in each of the vignettes that these students expect to make sense, reason, construct arguments, critique their classmates' reasoning, and engage in other central practices. Not seen in the vignettes is the work each teacher has done since the beginning of the year to create a classroom culture in which these behaviors become the norm.

WHAT IS THE CURRICULUM'S ROLE IN SUPPORTING THE MATH PRACTICES?

The three examples illustrate the first two principles that underlie how the Mathematical Practices are incorporated into the *Investigations 3* curriculum—that students are learning how to engage in mathematics through action and reflection and that the Practices occur in the context of mathematics content. Principles 3 and 4 emphasize how the Practices must be a deliberate focus of instruction and assessment. However, supporting these Mathematical Practices may be new to many teachers. The embedded professional development in *Investigations 3* addresses the questions: What are the Math Practices? What does it look like when students engage in them at a particular grade level in the context of particular content? How does the teacher support students to learn to engage in the Math Practices? How does the teacher assess students' progress?

Math Practice Essay from *Grade 1 Unit 7*
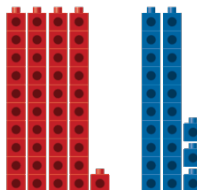

Math Practice Sidebars from *Grade 1 Unit 7*

Throughout the unit, Math Practice sidebars alert the teacher to opportunities to focus on these Practices with students.

Because *each* Practice requires focused instructional time within and across grades, each of the eight curriculum units highlights two of the Math Practices to be emphasized in instruction and assessment during that unit. In this way, students and teachers encounter a focus on each Practice twice during the school year. An essay, *Mathematical Practices in This Unit*, in the front matter of each unit, describes the two highlighted Practices with examples such as those in this paper.

Assessment Checklists help teachers keep track of students' progress. For example, in the Grade 1 example above, the two highlighted Practices in the unit are Math Practices 2 and 8. To assess MP2, teachers observe how students connect the numbers, the operation, and the equal sign in an equation to the quantities and actions in their representation. To assess MP8, they gather information about whether a student can apply the same strategy to multiple problems and how the student describes the steps of that strategy. These observations are recorded on an Assessment Checklist.

Assessment Checklist: Add Within 100 and 🔍 MP8

STUDENT	Adds Within 100			Can apply the same strategy to another pair of numbers 🔍 MP8	Describes the steps of their strategy (using the #s in the problem, using general terms) 🔍 MP8
	2-digit + a multiple of 10 (*Adding Tens*)	2-digit + 1-digit (*Roll Tens 2*)	2-digit + 2-digit (*Adding within 100, Three-in-a-Row on the 100 Chart, Race to the Top 2*)		
Toshi	(✓) from memory; +/–	counts on or uses a combo of 10	starts w/ larger #, counts on by 1s	yes	"I start with 43 and then 44, 45, 46, … 55."
Jacinta	(✓) from memory; +/–	mentally adds; uses combo of 10 35 + 7 = 35 + 5 + 2 = 40 + 2	Gathers/counts the tens, then the ones; knows 5 tens is 50	only strategy I have observed him using	"I put the tens together and that makes 50 and then 51, 52, 53, 54, 55."
Libby			"4 tens and 1 ten make 5 tens. 3 ones and 2 ones make 5 ones. 55."	uses this strategy when less than 10 ones. Got confused for 27 + 36 and counted on from 50.	"I add the tens and add the ones and that tells me the answer."
Vic	(✓) + from memory; pauses w/– but is correct	mostly counts on	starts w/ 1st #, counts on the 10s, then the 1s		"I count on the tens and count on the ones. It will work every time."

*Assessment Checklist from **Grade 1 Unit 7***

Once students have become familiar with a Math Practice, there are many opportunities to continue to apply that Practice within different content areas throughout the year. To facilitate their ongoing use, the Math Practice sidebars in all curriculum units alert the teacher to activities that particularly lend themselves to a focus on each of the eight Practices.

Key mathematical practices have always been embedded in the ways students are expected to solve problems in the *Investigations* curriculum. In *Investigations 3*, the authors have used the framework of the eight CCSS Mathematical Practices to highlight these important characteristics of the discipline of mathematics for students and teachers. These Practices, which are always employed in the context of mathematics content, are themselves an explicit focus of instruction, and are revisited within and across grades as students apply them in different content areas. The teacher tools—the Mathematical Practices essay in each unit, the sidebars, and the assessment checklists—help teachers to understand the Practices more deeply and to recognize the opportunities for student engagement.

References

Illustrative Mathematics. 2014, February 12. *Standards for Mathematical Practice: Commentary and Elaborations for K–5.* Tucson, AZ. [link: tinyurl.com/MPElaborations]

National Council of Teachers of Mathematics. 2000. *Principles and Standards for School Mathematics.* Reston VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics. 2014. *From Principles to Actions: Ensuring Mathematical Success for All.* Reston VA: National Council of Teachers of Mathematics.

National Research Council. 2001. *Adding It Up: Helping Children Learn Mathematics.* Washington, D.C.: The National Academies Press.

National Governors Association Center for Best Practices, Council of Chief State School Officers. 2010. *Common Core State Standards for Mathematics.* Washington, D.C.: National Governors Association Center for Best Practices, Council of Chief State School Officers.

DR. DEBORAH SCHIFTER
Dr. Deborah Schifter is a Principal Research Scientist at the Education Development Center in Waltham, MA. Deborah Schifter has worked as an applied mathematician; has taught elementary, secondary, and college level mathematics; and, since 1985, has been a mathematics teacher educator, curriculum writer, and educational researcher. She authored *Reconstructing Mathematics Education: Stories of Teachers Meeting the Challenge of Reform* and edited a two-volume anthology of teachers' writing, *What's Happening in Math Class?* With Virginia Bastable and Susan Jo Russell, she produced the professional development series, *Developing Mathematical Ideas*. She was a senior author on the second and third editions of *Investigations*, and over the last decade, her work has focused on early algebra elementary students' understanding of the operations.

DR. SUSAN JO RUSSELL
Dr. Susan Jo Russell is a Principal Scientist in the Education Research Collaborative at TERC. Susan Jo began her career in education as an elementary classroom teacher and math coach. She co-directed the original development and second edition of *Investigations in Number, Data and Space*, and is a senior author on *Investigations 3*. She also co-authored, with Deborah Schifter and others, the recent books, *Connecting Arithmetic to Algebra: Strategies for Building Algebraic Thinking in the Elementary Grades* (Heinemann, 2011) and *But Why Does It Work?: Mathematical Argument in the Elementary Classroom* (Heinemann, 2017). Her current work centers on early algebra, number and operations, mathematical argument, and how teachers can support students to engage in mathematical practices.

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